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THE SOLUTION OF ONE CLASS OF DUAL INTEGRAL EQUATIONS CONNECTED WITH THE MEHLER-FOCK TRANSFORM IN THE THEORY OF ELASTICITY AND MATHEMATICAL PHYSICS

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Dual integral equations with kernels containing spherical Legendre functions are examined. It is shown that these equations permit exact solution in quadratures. The proposed theory includes as a special case the theory of equations examined earlier which are connected with the Mehler-Fock transform and which are encountered in various applications, in particular in the solution of mixed boundary value problems in mathematical physics and in the theory of elasticity.

1. Equations of the following form are called dual equations connected with the integral transform of Mehler-Fock:

$$\int_0^\infty M(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0)$$

$$\int_0^\infty M(\tau)\, \omega(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = g(\alpha) \qquad (\alpha > \alpha_0) \tag{1.1}$$

here $P_\nu(z)$ is a spherical Legendre function with a complex index $\nu = -1/2 + i\tau$, $f(\alpha)$ and $g(\alpha)$ are given functions, $\omega(\tau)$ is the weight function ($\omega(\tau) > 0$, $\omega(\tau) \approx \tau$ for $\tau \to \infty$). Equations of this type are encountered in many applications; in particular, they play an important role in the solution of some mixed boundary value problems. Generalizations of Eqs. (1.1) are also examined. The kernels of these equations contain associated spherical functions.

At the present time a general theory of such equations does not exist, and a large part of results obtained in this area is related to equations of a special form which correspond to different selection of function $\omega(\tau)$ (see [1-6]). Thus, the following equations were studied

$$\int_0^\infty M(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0) \tag{1.2}$$

$$\int_0^\infty M(\tau) \frac{\operatorname{ch}\pi\tau}{\pi\,[P_{-1/2+i\tau}(0)]^2}\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = 0 \qquad (\alpha > \alpha_0)$$

They arise in the solution of boundary value problems with mixed boundary conditions given on the surface of a one-sheet hyperboloid of revolution [6]. Another interesting particular case is represented by the equations given in [1]

$$\int_0^\infty M(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0) \tag{1.3}$$

$$\int_0^\infty M(\tau)\, \tau \operatorname{th}\pi\tau P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = 0 \qquad (\alpha > \alpha_0)$$

The solution of these equations was the first step in the formulation of the theory of dual integral equations with kernels of Mehler-Fock. These equations find numerous applications in the solution of some contact problems, problems in the theory of elasticity, electrostatics, etc.

Equations (1.2) and (1.3) allow exact solutions in quadratures. The method of solution of these equations can be used at the same time to study equations of a more general type in which the function $\omega(\tau)$ coincides only asymptotically with the weight functions of equations which are under examination. For these cases the investigation of dual equations leads to the solution of Fredholm's integral equations of the second kind with a continuous kernel.

The purpose of this paper consists in the solution of dual integral equations

$$\int_0^\infty M(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0)$$

$$\int_0^\infty M(\tau)\, \omega_\mu(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = 0 \qquad (\alpha > \alpha_0) \tag{1.4}$$

$$\omega_\mu(\tau) = 4\pi^2 \Big[\operatorname{ch}\pi\tau\Gamma\Big(\frac{1}{4}+\frac{\mu}{2}+\frac{i\tau}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{\mu}{2}-\frac{i\tau}{2}\Big) \times \times\Gamma\Big(\frac{1}{4}-\frac{\mu}{2}+\frac{i\tau}{2}\Big)\Gamma\Big(\frac{1}{4}-\frac{\mu}{2}-\frac{i\tau}{2}\Big)\Big]^{-1} \tag{1.5}$$

Here $\Gamma(z)$ is the gamma function, μ is a parameter which assumes arbitrary real values (since $\omega_{-\mu}(\tau) = \omega_\mu(\tau)$, we can without limiting generality assume that $\mu \geqslant 0$).

Equations (1.2) correspond to the value $\mu = 0$, equations (1.3) – to the value $\mu = 1/2$. It is shown that Eqs. (1.4) belong to a class of equations which can be solved exactly. The proposed method of solution can be used to examine both particular cases investigated earlier from a single point of view. The developed theory makes it also possible to find solutions of equations with the weight function $\omega(\tau)$, which is close to $\omega_\mu(\tau)$.

2. Let us examine some discontinuous integrals containing spherical Legendre functions. Before going to the solution of dual integral equations of the type to be examined, we obtain two auxiliary equations which play an important role in the theory which follows. These equations have the form

$$\operatorname{ch} t \int_0^\infty \frac{\tau \operatorname{th}\pi\tau}{\omega_\mu(\tau)} F\Big(\frac{1}{4}+\frac{\mu}{2}+\frac{i\tau}{2},\ \frac{1}{4}+\frac{\mu}{2}-\frac{i\tau}{2},\ \frac{1}{2},\ -\operatorname{sh}^2 t\Big) P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau =$$

$$= \begin{cases} \dfrac{(\operatorname{ch} t)^{1-\mu}}{\sqrt{\operatorname{ch}^2 \alpha - \operatorname{ch}^2 t}} F\left(-\dfrac{\mu}{2}, \dfrac{\mu}{2}, \dfrac{1}{2}, \dfrac{\operatorname{ch}^2 t - \operatorname{ch}^2 \alpha}{\operatorname{ch}^2 t}\right) & (t < \alpha) \\ 0 & (t > \alpha) \end{cases} \tag{2.1}$$

$$\operatorname{sh} t \int_0^\infty \tau \operatorname{th} \pi\tau F\left(\frac{1}{4} + \frac{\mu}{2} + \frac{i\tau}{2}, \frac{1}{4} + \frac{\mu}{2} - \frac{i\tau}{2}, \frac{3}{2}, -\operatorname{sh}^2 t\right) P_{-1/2+i\tau}(\operatorname{ch} \alpha)\, d\tau =$$

$$= \begin{cases} 0 & (t < \alpha) \\ \dfrac{(\operatorname{ch} t)^{1-\mu}}{\sqrt{\operatorname{ch}^2 t - \operatorname{ch}^2 \alpha}} F\left(\dfrac{1-\mu}{2}, \dfrac{\mu - 1}{2}, \dfrac{1}{2}, \dfrac{\operatorname{ch}^2 t - \operatorname{ch}^2 \alpha}{\operatorname{ch}^2 t}\right) & (t > \alpha) \end{cases} \tag{2.2}$$

here $F(a, b, c, z)$ is a hypergeometric function. The first of these equations is valid for $0 \leqslant \mu \leqslant {}^1/_2$, the second for any $\mu \geqslant 0$.

Equations (2.1) and (2.2) apparently will be new, and their derivation is based on relatively complicated considerations. A brief development will be presented which alows to obtain Eq. (2.1). Taking advantage of well known transformation for hypergeometric function

$$F(a, b, {}^1/_2, z) = (1 - z)^{-a} \frac{\sqrt{\pi}\Gamma(b - a)}{\Gamma({}^1/_2 - a)\Gamma(b)} F\left(a, \frac{1}{2} - b, 1 + a - b, \frac{1}{1 - z}\right) +$$

$$+ (1 - z)^{-b} \frac{\sqrt{\pi}\Gamma(a - b)}{\Gamma({}^1/_2 - b)\Gamma(a)} F\left(b, \frac{1}{2} - a, 1 - a + b, \frac{1}{1 - z}\right)$$

and taking $\nu = i\tau$, we can represent the integral in the left side of (2.1) in the form

$$J = \frac{(\operatorname{ch} t)^{1/2-\mu}}{4i\sqrt{\pi}} \int_{-i\infty}^{i\infty} \frac{\nu}{\Gamma(1 + \nu)} \Gamma\left(\frac{1}{4} + \frac{\mu}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{1}{4} - \frac{\mu}{2} + \frac{\nu}{2}\right) \times$$

$$\times (\operatorname{ch} t)^{-\nu} F\left(\frac{1}{4} + \frac{\mu}{2} + \frac{\nu}{2}, \frac{1}{4} - \frac{\mu}{2} + \frac{\nu}{2}, 1 + \nu, \frac{1}{\operatorname{ch}^2 t}\right) P_{\nu - 1/2}(\operatorname{ch} \alpha)\, d\nu \tag{2.3}$$

Initially it is assumed that $t > \alpha$. The integrand in (2.3) represents a meromorphic function with poles in points $\nu = -2n - {}^1/_2 \pm \mu$ $(n = 0, 1, 2, \ldots)$. For the condition $0 \leqslant \mu \leqslant {}^1/_2$ these poles are located to the left of the imaginary axis (for $\mu = {}^1/_2$ the point $\nu = 0$ is a removable singular point). Therefore, complementing the contour of integration with an arc of the circle of the large radius located in the half-plane $\operatorname{Re} \nu > 0$, and noting on the basis of well known asymptotic equations, for $|\nu| \to \infty$, $|\arg \nu| \leqslant {}^1/_2 \pi$ the integrand tends to zero as $2^{1/2}(\pi\nu \operatorname{sh}\alpha)^{-1/2} e^{-(t-\alpha)\nu}$, we obtain

$$J|_{t > \alpha} = 0$$

For the evaluation of the integral at $t < \alpha$ the following functional relationship is utilized

$$\pi \operatorname{tg} \pi\nu P_{\nu - 1/2}(z) = Q_{-\nu - 1/2}(z) - Q_{\nu - 1/2}(z)$$

and J is represented as the sum of two integrals (*)

$$J = \frac{(\operatorname{ch} t)^{1/2-\mu}}{4i\pi\sqrt{\pi}} \left\{ \int_{-i\infty}^{i\infty} \frac{\nu \operatorname{ctg} \pi\nu}{\Gamma(1 - \nu)} \Gamma\left(\frac{1}{4} + \frac{\mu}{2} - \frac{\nu}{2}\right) \Gamma\left(\frac{1}{4} - \frac{\mu}{2} - \frac{\nu}{2}\right) \times \right. \tag{2.4}$$

*) In the first integral the substitution of ν by $-\nu$ has been performed.

(cont.)

$$\times (\operatorname{ch} t)^{\nu} F\left(\frac{1}{4}+\frac{\mu}{2}-\frac{\nu}{2},\ \frac{1}{4}-\frac{\mu}{2}-\frac{\nu}{2},\ 1-\nu,\ \frac{1}{\operatorname{ch}^2 t}\right) Q_{\nu-1/2}(\operatorname{ch}\alpha)\, d\nu -$$

$$-\int_{-i\infty}^{i\infty} \frac{\nu \operatorname{ctg}\pi\nu}{\Gamma(1+\nu)} \Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{4}-\frac{\mu}{2}+\frac{\nu}{2}\right)\times$$

$$\times (\operatorname{ch} t)^{-\nu} F\left(\frac{1}{4}+\frac{\mu}{2}+\frac{\nu}{2},\ \frac{1}{4}-\frac{\mu}{2}+\frac{\nu}{2},\ 1+\nu,\ \frac{1}{\operatorname{ch}^2 t}\right) Q_{\nu-1/2}(\operatorname{ch}\alpha)\, d\nu\Bigg\}$$

With the assumption that $0 \leqslant \mu \leqslant 1/2$ singular points in the first integral located in the half-plane $\operatorname{Re}\nu > 0$ will be poles $\nu = n$ $(n = 1, 2, \ldots)$ and $\nu = 2n + 1/2 \pm \mu$ $(n = 0, 1, 2, \ldots)$, in the second integral, poles $\nu = n$ $(n = 1, 2, \ldots)$. Supplementing the contour of integration to a closed contour and taking into account the asymptotic behavior of integrands for $|\nu| \to \infty$, $|\arg\nu| \leqslant 1/2\,\pi$, we obtain using the residue theory (*)

$$J = \frac{(\operatorname{ch} t)^{1-\mu}}{\sqrt{\pi}} \left\{ \sum_{n=0}^{\infty} (-1)^n \frac{(2n-\mu+1/2)\,\Gamma(\mu-n)\operatorname{tg}\pi\mu}{n!\,\Gamma(-2n+\mu+1/2)} \times \right.$$

$$\times (\operatorname{ch} t)^{2n-\mu} F(-n,\ \mu-n,\ -2n+\mu+1/2,\ \operatorname{sch}^2 t)\, Q_{2n-\mu}(\operatorname{ch}\alpha) -$$

$$-\sum_{n=0}^{\infty} (-1)^n \frac{(2n+\mu+1/2)\,\Gamma(-\mu-n)\operatorname{tg}\pi\mu}{n!\,\Gamma(-2n-\mu+1/2)} \times$$

$$\left. \times (\operatorname{ch} t)^{2n+\mu} F(-n,\ -\mu-n,\ -2n-\mu+1/2,\ \operatorname{sch}^2 t)\, Q_{2n+\mu}(\operatorname{ch}\alpha)\right\} \qquad (2.5)$$

The summation in the right side of Eq. (2.5) can be carried out on the basis of general theory for series of this type which is due to Schäfke [7]. On the basis of this theory we find

$$J\,|_{t<\alpha} = \frac{(\operatorname{ch} t)^{1-\mu}}{\sqrt{\operatorname{ch}^2\alpha - \operatorname{ch}^2 t}} F\left(-\frac{\mu}{2},\ \frac{\mu}{2},\ \frac{1}{2},\ \frac{\operatorname{ch}^2 t - \operatorname{ch}^2\alpha}{\operatorname{ch}^2 t}\right)$$

The proposed method of evaluation of integral (2.1) is also applicable for $\mu > 1/2$; however, in this case the distribution of singular points in the plane of the complex variable turns out to be somewhat different. This leads to the appearance of additional terms in the right side of Eq. (2.1). The number of these terms depends on the value of parameter μ. Without writing the explicit expressions for these terms, we note that they will be continuous functions of the variable t in the interval $(0, \infty)$.

Equation (2.2) is proven in an analogous manner. Here for this work it is sufficient to establish the validity of the equation for $t < \alpha$.

3. Let us construct the solution of dual integral equations for values $0 \leqslant \mu \leqslant 1/2$ Let us examine dual integral equations

$$\int_0^{\infty} M(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0) \qquad (3.1)$$

$$\int_0^{\infty} M(\tau)\, \omega_\mu(\tau)\, P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau = 0 \quad (\alpha > \alpha_0) \qquad (3.2)$$

here $\omega_\mu(\tau)$ has the significance indicated in Eq. (1.5), $f(\alpha)$ is a given continuously differentiable function.

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*) Residues related to points $\nu = n$ $(n = 1, 2, \ldots)$ are cancelled in the computation.

In the solution of Eqs. (3.1) and (3.2) we shall assume initially that the parameter μ belongs to the interval $(0, {}^1/_2)$. We shall seek the solution of equations under examination in the form (3.3)

$$M(\tau) = \frac{2\tau \operatorname{th} \pi\tau}{\pi\omega_\mu(\tau)} \int_0^{\alpha_0} \varphi(t) \operatorname{ch} t F\left(\frac{1}{4} + \frac{\mu}{2} + \frac{i\tau}{2}, \frac{1}{4} + \frac{\mu}{2} - \frac{i\tau}{2}, \frac{1}{2}, -\operatorname{sh}^2 t\right) dt$$

here $\varphi(t)$ is an unknown function which is continuous together with its first derivative in the interval $(0, \alpha_0)$.

Taking into consideration the relationship

$$F(a, b, {}^1/_2, -z^2) = \frac{d}{dz} zF(a, b, {}^3/_2, -z^2)$$

and integrating (3.3) by parts, we have

$$M(\tau) = \frac{2\tau \operatorname{th} \pi\tau}{\pi\omega_\mu(\tau)} \Big\{\varphi(\alpha_0) \operatorname{sh} \alpha_0 F\Big(\frac{1}{4} + \frac{\mu}{2} + \frac{i\tau}{2}, \frac{1}{4} + \frac{\mu}{2} - \frac{i\tau}{2}, \frac{3}{2}, -$$

$$- \operatorname{sh}^2\alpha_0\Big) - \int_0^{\alpha_0} \varphi'(t) \operatorname{sh} t F\left(\frac{1}{4} + \frac{\mu}{2} + \frac{i\tau}{2}, \frac{1}{4} + \frac{\mu}{2} - \frac{i\tau}{2}, \frac{3}{2}, -\operatorname{sh}^2 t\right) dt\Big\}$$

If the last expression is substituted into (3.2) and if advantage is taken of the value of integral (2.2), the Eq. (3.2) will be satisfied as an identity.

Substituting (3.3) into (3.1) and taking into account Eq. (2.1), we arrive at the integral equation of the first kind for the function $\varphi(t)$

$$\int_0^{\alpha} \varphi(t) \frac{(\operatorname{ch} t)^{1-\mu}}{\sqrt{\operatorname{ch}^2\alpha - \operatorname{ch}^2 t}} F\left(-\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{2}, \frac{\operatorname{ch}^2 t - \operatorname{ch}^2\alpha}{\operatorname{ch}^2 t}\right) dt = \frac{\pi}{2} f(\alpha) \quad (3.4)$$

$$(0 \leqslant \alpha \leqslant \alpha_0)$$

Equation (3.4) takes a simpler form if we set

$$x = \operatorname{ch}^2\alpha, \quad y = \operatorname{ch}^2 t, \quad \psi(y) = \frac{\sqrt{\pi}}{2} \frac{y^{-{}^1/_2\mu}}{\sqrt{y-1}} \varphi(\operatorname{ar ch} \sqrt{y}), \quad g(x) = \frac{\pi}{2} f(\operatorname{ar ch} \sqrt{x}) \quad (3.5)$$

After a change of variables we obtain

$$\int_1^x \frac{\psi(y)}{\sqrt{\pi}\sqrt{x-y}} F\left(-\frac{\mu}{2}, \frac{\mu}{2}, \frac{1}{2}, 1 - \frac{x}{y}\right) dy = g(x) \qquad (x \geqslant 1) \quad (3.6)$$

The last equation belongs to a class of integral equations recently investigated by Love [8]. This class includes many equations of practical importance, in particular Abel's integral equation which plays an important role in the solution of dual integral equations (1.2) and (1.3).

According to Love's theory the solution of Eq. (3.6) has the form

$$\psi(y) = y^{{}^1/_2\mu} \frac{d}{dy} y^{-{}^1/_2\mu} \int_1^y \frac{g(x)}{\sqrt{\pi}\sqrt{y-x}} F\left(\frac{\mu}{2}, 1 - \frac{\mu}{2}, \frac{1}{2}, 1 - \frac{x}{y}\right) dx \quad (3.7)$$

Returning to the initial variables, we find (3.8)

$$\varphi(t) = (\operatorname{ch} t)^{2\mu-1} \frac{d}{dt} (\operatorname{ch} t)^{-\mu} \int_0^t \frac{f(\alpha) \operatorname{ch} \alpha \operatorname{sh} \alpha}{\sqrt{\operatorname{ch}^2 t - \operatorname{ch}^2 \alpha}} F\left(\frac{\mu}{2}, 1 - \frac{\mu}{2}, \frac{1}{2}, 1 - \frac{\operatorname{ch}^2\alpha}{\operatorname{ch}^2 t}\right) d\alpha$$

After determination of $\varphi(t)$ the solution of dual integral equations (3.1) and (3.2) is obtained in quadratures using Eq. (3. 3). In this manner Eq. (3. 8) gives the solution of the formulated problem.

This theory contains as particular cases results obtained earlier with respect to dual integral equations (1. 2) and (1. 3).

Computations in Sect. 3 have a formal character. However, with limitations placed on function $f(\alpha)$ above, Eq. (3. 4) allows a solution in the class of continuously differentiable functions. This solution is determined from Eq. (3. 8). Based on this we can show that Eq. (3. 3) gives a continuous solution of dual equations (3. 1) and (3. 2).

As an example let us examine the case $f(\alpha) = 1$. For a right side of this kind the desired function $\varphi(t)$ can be expressed in a closed form through a hypergeometric function. Utilizing Eq. (3. 8), after some subsequent computations we obtain

$$\varphi(t) = (\operatorname{ch} t)^{2\mu-1} \frac{d}{dt} (\operatorname{ch} t)^{-\mu} \operatorname{sh} t F\left(\frac{\mu}{2}, 1 - \frac{\mu}{2}, \frac{3}{2}, \operatorname{th}^2 t\right)$$

4. The method proposed above for the solution of dual equations (3. 1) and (3. 2) in principle can be generalized to the case $0 \leqslant \mu < \infty$ however, the entire theory assumes in this case a somewhat cumbersome form. For this reason we limit ourselves to a brief review of this theory without presenting the explicit form of the solution and without details of computations.

As before, the solution of dual equations is sought in the form (3. 3). In this case on the basis of (2. 2) the homogeneous equation (3. 2) is satisfied as an identity. Substitution of (3. 3) into the inhomogeneous equation (3. 1) leads (with consideration of the correspondingly modified form of integral (2. 1)) to an integral equation for the function $\varphi(t)$ which differs from Eq. (3. 4) because its right side contains in addition to the given function $f(\alpha)$ a supplementary term of the form

$$\sum_{n=0}^{N} c_n P_{2n-\mu}(\operatorname{ch} \alpha)$$

Coefficients c_n represent integrals of products of the desired function $\varphi(t)$ with some known functions of the variable t over the interval $(0, \alpha_0)$. The number N is connected with the parameter μ by the relationship $N = [{}^1/_2\, \mu - {}^1/_4]$. The number of terms in this sum is therefore always finite (*). After solution of the integral equation, using the inversion formula (3. 8), the determination of unknown constants c_n is reduced to the solution of a system of linear algebraic equations.

As an example which illustrates the peculiarities of the theory arising at $\mu > {}^1/_2$, let us examine the solution of the following dual integral equations

$$\int_0^\infty M(\tau) P_{-1/2+i\tau}(\operatorname{ch} \alpha)\, d\tau = f(\alpha) \qquad (0 \leqslant \alpha < \alpha_0) \tag{4.1}$$

$$\int_0^\infty M(\tau)\, \omega_1(\tau) P_{-1/2+i\tau}(\operatorname{ch} \alpha)\, d\tau = 0 \quad (\alpha > \alpha_0) \tag{4.2}$$

which correspond to a value of $\mu = 1$ and which occur in practical applications. In

*) In particular, for the interval $0 \leqslant \mu \leqslant {}^1/_2$ the sum turns out to be empty, and the additional term is absent (for $\mu = {}^1/_2$ the coefficient $c_0 = 0$).

this case the discontinuous integral (2.1) has the form

$$\operatorname{ch} t \int_0^\infty \frac{\tau \operatorname{th} \pi\tau}{\omega_1(\tau)} F\left(\frac{3}{4} + \frac{i\tau}{2}, \frac{3}{4} - \frac{i\tau}{2}, \frac{1}{2}, -\operatorname{sh}^2 t\right) P_{-1/2+i\tau}(\operatorname{ch}\alpha)\, d\tau =$$

$$= \begin{cases} \theta(t) - \operatorname{sch} t\,(t < \alpha) \\ \quad - \operatorname{sch} t\,(t > \alpha) \end{cases}, \qquad \theta(t) = \frac{\operatorname{ch}\alpha}{\operatorname{ch} t \sqrt{\operatorname{ch}^2 \alpha - \operatorname{ch}^2 t}} \tag{4.3}$$

We note that the first term $\theta(t)$ in the right side of (4.3) can be obtained from (2.1) if $\mu = 1$ is taken there. The second term corresponds to the additional residue in the strip $\nu = {}^1/_2$. This residue arises in the evaluation of integrals (2.3) and (2.4).

If, according to (3.3), the solution of Eqs. (4.1) and (4.2) is sought in the form

$$M(\tau) = \frac{2\tau \operatorname{th} \pi\tau}{\pi\omega_1(\tau)} \int_0^{\alpha_0} \varphi(t) \operatorname{ch} t\, F\left(\frac{3}{4} + \frac{i\tau}{2}, \frac{3}{4} - \frac{i\tau}{2}, \frac{1}{2}, -\operatorname{sh}^2 t\right) dt \tag{4.4}$$

then Eq. (4.2) is satisfied as an identity, while for the determination of $\varphi(t)$ the following integral equation is obtained

$$\operatorname{ch}\alpha \int_0^\alpha \frac{\varphi(t)\, dt}{\operatorname{ch} t \sqrt{\operatorname{ch}^2 \alpha - \operatorname{ch}^2 t}} = \frac{\pi}{2} f(\alpha) + c \tag{4.5}$$

The kernel of this equation coincides with the kernel of Eq. (3.4) for $\mu = 1$. The unknown additive constant c is related to the desired function through the relationship

$$c = \int_0^{\alpha_0} \frac{\varphi(t)}{\operatorname{ch} t}\, dt \tag{4.6}$$

Solving Eq. (4.5) according to the scheme indicated above, we find

$$\varphi(t) = \operatorname{ch} t \frac{d}{dt} \int_0^t \frac{f(\alpha) \operatorname{sh} \alpha\, d\alpha}{\sqrt{\operatorname{ch}^2 t - \operatorname{ch}^2 \alpha}} + \frac{2c}{\pi} \tag{4.7}$$

Multiplying the last equation by $(\operatorname{ch} t)^{-1}$ and integrating over the interval $(0, \alpha_0)$, we obtain for the determination of c a linear equation, from which follows

$$c = \frac{1}{1 - 2\pi^{-1} \operatorname{arctg} \operatorname{sh} \alpha_0} \int_0^{\alpha_0} \frac{f(\alpha) \operatorname{sh} \alpha\, d\alpha}{\sqrt{\operatorname{ch}^2 \alpha_0 - \operatorname{ch}^2 \alpha}} \tag{4.8}$$

The mathematical method developed in this paper can also be used for the investigation of dual equations of Mehler-Fock when $\omega(\tau)$ is close to $\omega_\mu(\tau)$. Application of the usual procedure presents the possibility to obtain the solution of these equations in the form of quadratures containing the supplementary function $\varphi(t)$ which satisfies an integral equation of the second kind with a continuous kernel. For the particular cases $\mu = 0$ and $\mu = {}^1/_2$ the corresponding calculations were carried out in [1, 2 and 6].

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ON MAGNETOPLASTIC FLOW

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The second law of thermodynamics is used: (a) to derive the flow equations for a magnetoplastic medium; (b) to investigate in detail the magnetoplastic flow of a long thick-walled pipe; (c) to consider the flow of a pipe acted on by a nonpenetrating field; (d) to find the conditions under which the "infrozen" magnetic field facilitates plastic flow.

Magnetic fields capable of producing stresses in excess of the yield stress of metals have been achieved [1]. If the conductivity of the metal is sufficiently high, then the presence of infrozen magnetic lines of force [2] results in interaction between the plastic flow and the magnetic field. This is what constitutes magnetoplastic flow. Magnetoplastic effects are manifested if the magnetic pressure is of the order of the yield stress of the material, i.e. if $^1/_8 H^2/\pi \approx k$. In the case of hard coppers ($k \approx 40$ kg/mm^2) the field intensity required is $H \approx 300$ kOe; for hard steels ($k \approx 100$ kg/mm^2) $H \approx 450$ kOe.

1. Let us make use of the second law of thermodynamics. The law of conservation of the energy W in some volume V can be written as [3]

$$dW = \delta A + d_e W \tag{1.1}$$

Here A is the work done by the external forces; $d_e W$ is the energy influx through the surface.

The work done per unit time can be resolved [4] into the work done by the external surface forces $\partial_e A \,/\, \partial t$

$$\frac{\partial_e A}{\partial t} = \oint_\Omega v_i \sigma_{ij}^\circ \, d\Omega_j \tag{1.2}$$

and that done by the external body forces (the Lorentz forces $\partial_i A \,/\, \partial t$),

$$\frac{\partial_i A}{\partial t} = \int_V \left(\frac{\mathbf{j}}{c} \times \mathbf{H}\right) \mathbf{v} \, dV \tag{1.3}$$